Basic Algorithms for Periodic-Linear Inequalities and Integer Polyhedra

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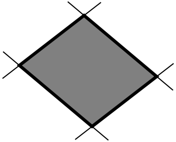
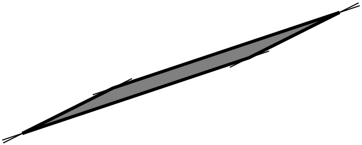
Omega's nightmare

$$\begin{aligned} 3 \le\ & 11x + 13y \le 21 \\ -8 \le\ & 7x - 9y \le 6 \end{aligned}$$



LS(N)

$$\begin{aligned} 2 \le\ & 3y - x \le 5 \\ 1 - N \le\ & 2y - x \le 1 \end{aligned}$$



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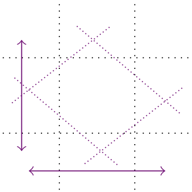

- empty (no integer points)
- non-empty rational projections

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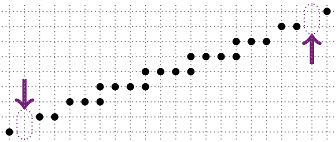
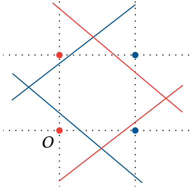


- holes
- "fuzzy" vertices
- periodic y-span

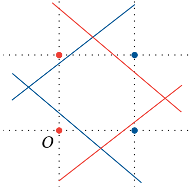


Omega's nightmare

$$-11x + 3 \leq 13y \leq -11x + 21$$
$$7x - 6 \leq 9y \leq 7x + 8$$

Fourier-Motzkin variable elimination (of y)

$$\begin{aligned} 9 \cdot (-11x + 3) &\leq 13 \cdot (7x + 8) \\ 13 \cdot (7x - 6) &\leq 9 \cdot (-11x + 21) \end{aligned} \quad \Rightarrow \quad -77 \leq 190x \leq 267$$



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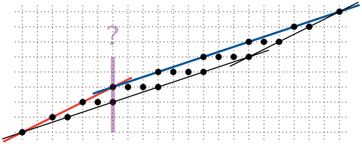
What went wrong? Loose bounds…

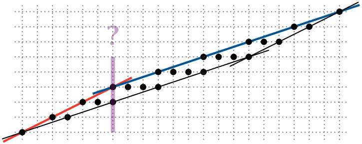
- A tight bound would be: $9y \leq 7x + 8 - (7x + 8) \bmod 9$
- Combinations accumulate and amplify the "slack"

$$-77 + \underbrace{\left[9(12 - (2 - 11x) \bmod 13) + 13((7x + 8) \bmod 9)\right]}_{\text{up to } 2 \cdot 9 \cdot 13 - 9 - 13 = 212} \leq 190x$$



$$\mathrm{LS}(N)$$
$$x + 2 \le 3y \le x + 5$$
$$x + 1 - N \le 2y \le x + 1$$

```
for 1 <= x <= 3N+7
    for max(⌈(x+2)/3⌉, ⌈(x+1−N)/2⌉) <= y <= min(⌊(x+5)/3⌋, ⌊(x+1)/2⌋)
        exec S(x,y)
```



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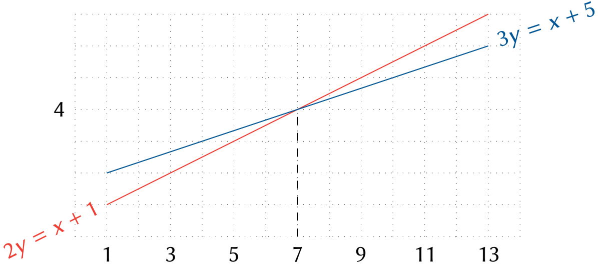
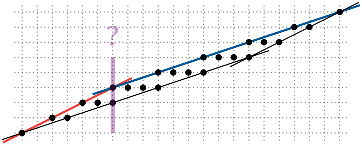
Which (upper) bound is effective where?





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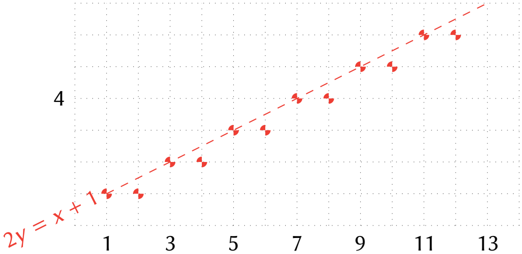

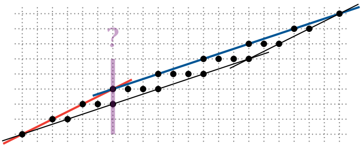

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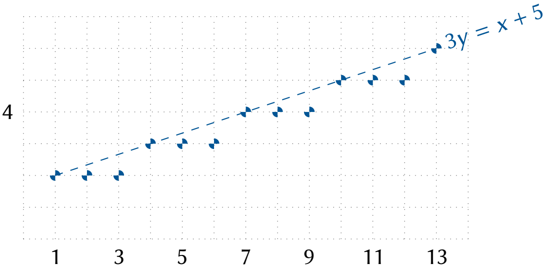

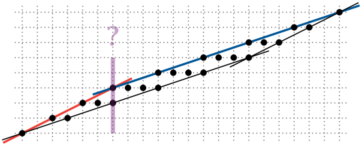

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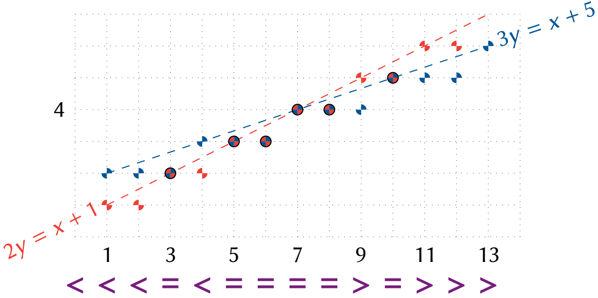

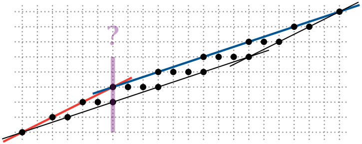

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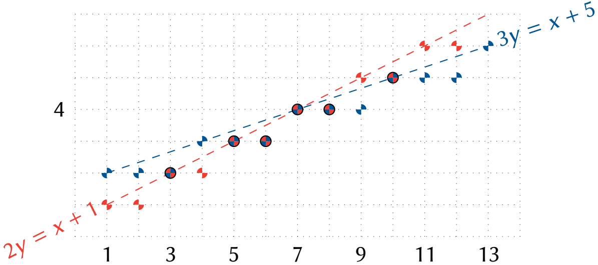


$$3 \cdot (x + 1 - (x + 1) \bmod 2) \quad \le\ge \quad 2 \cdot (x + 5 - (x + 5) \bmod 3)$$

1. Tightening bounds
   - find a workable modulo representation
   - define precise combinations
2. The Omicron Test
   - Fourier-Motzkin-like decision procedure
   - correct and complete
3. Polyhedron Decomposition
   - via affine unswitching
   - applications: transformation, projection, optimization

Strategy

- Be Radically Integral
  - no $\lfloor - \rfloor$ or $\lceil - \rceil$, no inexact division
  - no loose bound!
- Focus on Representation, not on Algorithms
  - find a set of elementary operations
  - algorithms should simply repeat while possible

A *periodic number* is a collection of numbers indexed by the congruence class of an expression:

$$\left\langle v_0, v_1, \ldots, v_{\pi-1} \right\rangle_x^\pi = \begin{cases} v_0 & \text{if } x \equiv 0 \bmod \pi \\ v_1 & \text{if } x \equiv 1 \bmod \pi \\ \vdots & \\ v_{\pi-1} & \text{if } x \equiv (\pi - 1) \bmod \pi \end{cases}$$

Essentially a notation, with useful operations:

Rotation	$\langle v_0, v_1, \ldots \rangle_{x+1}^\pi = \langle v_1, \ldots, v_0 \rangle_x^\pi$
Division	$\langle v_0, \ldots \rangle_{cx}^\pi = {}^i\left\langle \ldots, v_{(ci \bmod \pi)}, \ldots \right\rangle_x^{\pi / \gcd(\pi, c)}$
Distribution	$\langle v_0, \ldots \rangle^\alpha_{\langle w_0, \ldots, w_{\beta-1} \rangle_x^\beta} = {}^i\left\langle \ldots, \langle v_0, \ldots \rangle_{w_i}^\alpha, \ldots \right\rangle_x^\beta$
Separation	$\langle v_0, \ldots, v_{\alpha-1} \rangle_{x+y}^\alpha = {}^i\left\langle \ldots, \langle v_0, \ldots, v_{\alpha-1} \rangle_{i+y}^\alpha, \ldots \right\rangle_x^\alpha$

Modulos: for any expression X

$$X \bmod \pi = \langle 0, 1, \ldots, \pi - 1 \rangle_X^\pi$$

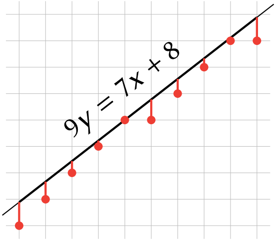
The maximal multiple of π less than or equal to X

$$X - X \bmod \pi = X - \langle 0, 1, \ldots, \pi - 1 \rangle_X^\pi$$

Tightened linear bounds have:

- a sharp linear part
- a periodic correction

$$\begin{aligned} 9y &\le 7x + 8 - (7x + 8) \bmod 9 \\ &\le 7x + 8 - \langle 0, \ldots, 8 \rangle_{7x+8}^9 \\ &\le 7x + 8 - \langle 8, 6, 4, 2, 0, 7, 5, 3, 1 \rangle_x^9 \end{aligned}$$



Works for any number of variables:

$$\langle v_0, v_1, v_2 \rangle_{2x+6y+5z-1}^3 = \left\langle \left\langle \langle v_2, v_1, v_0 \rangle_x^3 \right\rangle_y^1, \left\langle \langle v_1, v_0, v_2 \rangle_x^3 \right\rangle_y^1, \left\langle \langle v_0, v_2, v_1 \rangle_x^3 \right\rangle_y^1 \right\rangle_z^3$$

Periodic-linear expressions (PLEs) in *normal* form over $[x_1, \ldots, x_n]$:

$$\left(\sum_{i=1}^{n} a_i x_i\right) + \left\langle \cdots, \quad \left\langle \cdots \langle \cdots \rangle_{x_1}^{\pi_1} \cdots \right\rangle_{x_{n-1}}^{\pi_{n-1}}, \quad \cdots \right\rangle_{x_n}^{\pi_n}$$

or in *simplified* normal form

$$a_n x_n + \left\langle \cdots, \quad a_{n-1} x_{n-1} + \langle \cdots \rangle_{x_{n-1}}^{\pi_{n-1}}, \quad \cdots \right\rangle_{x_n}^{\pi_n}$$

If X and Y are PLEs, n an integer, then:

$$nX, \quad (X + Y), \quad (X \bmod \pi), \quad X[Y/x_k] \quad \text{are all PLEs}$$

Periodic-linear inequalities are PLEs compared to zero:

$$a_n x_n + \langle X_0, \ldots \rangle_{x_n}^{\pi_n} \geq 0 \qquad \text{with } X_0, \ldots \text{ PLEs over } [x_1, \ldots, x_{n-1}]$$

LS(N) over $[N, x, y]$ with tightened inequalities:

$$x + \langle 3,2,4\rangle_x \;\le\; 3y \;\le\; x + \langle 3,5,4\rangle_x$$
$$x - N + \Big\langle \langle 2,1\rangle_N , \langle 1,2\rangle_N \Big\rangle_x \;\le\; 2y \;\le\; x + \langle 0,1\rangle_x$$

and over $[N, x]$ after combination

$$\langle\langle 2,1\rangle_N , \langle 0,1\rangle_N\rangle_x - N \le 0 \qquad 0 \le \langle 3,0,3\rangle_x$$
$$\langle 6,1,8,3,4,5\rangle_x \le x \qquad x \le 3N + \begin{pmatrix}\langle 0,3\rangle_N , \langle 7,4\rangle_N , \langle 2,5\rangle_N , \\ \langle 3,0\rangle_N , \langle 4,7\rangle_N , \langle 5,2\rangle_N \end{pmatrix}_x^6$$

Categories:

- *linear*: $3y \le \Big\langle x + \langle 3,5,4\rangle_x \Big\rangle_y^1$
- *periodic*: $\Big\langle \langle 2,1\rangle_N , \langle 0,1\rangle_N \Big\rangle_x - N \le 0x$
- *mixed*: $\Big\langle 6,1,8,3,4,5 \Big\rangle_x \le x$

Given a (potentially loose) periodic-linear inequality over $[\dots, x_n]$:

$$a_n x_n \leq \langle X_0, \dots \rangle_{x_n}^{\pi_n} \qquad \text{or} \qquad \langle X_0, \dots \rangle_{x_n}^{\pi_n} \leq a_n x_n$$

the following inequality is an *equivalent* tight bound

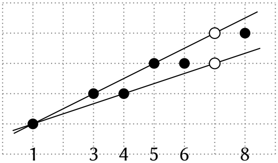
$$a_n x_n \leq {}^{i}\Big\langle \dots,\ X_i - \Big\langle 0, 1, \dots, a_n\pi_n - 1 \Big\rangle_{X_i - i a_n}^{a_n \pi_n},\ \dots \Big\rangle_{x_n}^{\pi_n}$$

→ the rhs is a multiple of a_n *for all* phases of x_n modulo π_n (and all phases of the other variables)

Mixed tight bounds are "fuzzy"

$$x + \langle 3, 2, 4\rangle_x \le 3y \quad 2y \le x + \langle 0, 1\rangle_x$$

$$\Rightarrow \qquad \langle 6, 1, 8, 3, 4, 5\rangle_x \le x$$



Disjoin turns a *mixed* bound into a disjunction of *linear* bounds: it computes a *major* bound plus *outliers*

$$\langle 6, 1, 8, 3, 4, 5\rangle_x \le x \qquad \Rightarrow \qquad (x = 1) \vee (3 \le x)$$

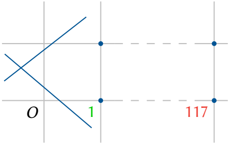
$$x \le 3N + \left\langle \begin{matrix} \langle 0, 3\rangle_N, \langle 7, 4\rangle_N, \langle 2, 5\rangle_N, \\ \langle 3, 0\rangle_N, \langle 4, 7\rangle_N, \langle 5, 2\rangle_N \end{matrix} \right\rangle_x^6 \qquad \Rightarrow \qquad (x \le 3N + 5) \vee (x = 3N + 7)$$

Omega's nightmare (left corner)

$$\frac{-11x + \langle \ldots \rangle_x^{13} \le 13y \qquad 9y \le 7x + \langle \ldots \rangle_x^{9}}{\langle 117, 73, 29, -15, \ldots, -29, 44\rangle_x^{117} \le 190x}$$

$$\Rightarrow \qquad \langle {\color{red}117}, {\color{green}1}, 2, 3, \ldots, 115, 116\rangle_x^{117} \le x$$

$$\Rightarrow \qquad {\color{green}1} \le x$$



Fourier-Motzkin elimination on $\mathbb{Q}$ relies on an equivalence:

$$f_l(x_1,\ldots,x_{n-1}) \le ax_n \qquad bx_n \le f_u(x_1,\ldots,x_{n-1})$$

$$\mathbb{Z},\mathbb{Q}\Big\downarrow \quad \Big\uparrow\mathbb{Q}$$

$$b \cdot f_l(x_1,\ldots,x_{n-1}) \le a \cdot f_u(x_1,\ldots,x_{n-1})$$

Restoring completeness on $\mathbb{Z}$ by tightening

loose bounds	$(4 \le 3x)$	$L \le ax$	$bx \le U$	$(3x \le 5)$
		$\updownarrow$	$\updownarrow$	
tight bounds	$(6 \le 3x)$	$L' \le ax$	$bx \le U'$	$(3x \le 3)$
		$\updownarrow$		
combination		$bL' \le aU'$	$(6 \le 3)$	

$\rightarrow$ keep bounds tight at all times

Combining *mixed* bounds may *not* eliminate the variable

$$\langle 6,1,8,3,4,5\rangle_x \le x \qquad x \le 3N + \left\langle \begin{matrix} \langle 0,3\rangle_N, \langle 7,4\rangle_N, \langle 2,5\rangle_N, \\ \langle 3,0\rangle_N, \langle 4,7\rangle_N, \langle 5,2\rangle_N \end{matrix} \right\rangle_x^6$$

$$\Rightarrow \quad \left\langle \begin{matrix} \langle 2,1\rangle_N, \langle -2,-1\rangle_N, \langle 2,1\rangle_N, \\ \langle 0,1\rangle_N, \langle 0,-1\rangle_N, \langle 0,1\rangle_N \end{matrix} \right\rangle_{\boxed{x}}^6 \le N$$

→ apply Disjoin, and fork the system (if needed)

There is no way to combine *periodic* bounds

$$\langle \langle 2,1\rangle_N, \langle 0,1\rangle_N\rangle_x^2 \le N \qquad \Rightarrow \begin{cases} x = 2x' + 0 \;\wedge\; \langle 2,1\rangle_N \le N \\ x = 2x' + 1 \;\wedge\; \langle 0,1\rangle_N \le N \end{cases}$$

→ *splinter* the system and change variables

$$\begin{array}{c} \hline 3 - 11x \leq 13y \leq 21 - 11x \\ 7x - 6 \leq 9y \leq 7x + 8 \\ \hline \end{array}$$

$\Rightarrow$ tighten+combine

$\downarrow$

$$\begin{array}{c} \hline \langle 117, 73, 29, \ldots, 44\rangle_x^{117} \leq 190x \\ 190x \leq \langle 117, 73, 146 \ldots, 161\rangle_x^{117} \\ \hline \end{array}$$

$\Rightarrow$?

$\Rightarrow$ tighten+disjoin

$\downarrow$

$$\begin{array}{c} \hline 1 \leq x \leq 0 \\ \hline \end{array}$$

$\Rightarrow$ combine

$\downarrow$

false

$$3 - 11x \leq 13y \leq 21 - 11x$$
$$7x - 6 \leq 9y \leq 7x + 8$$

$\Rightarrow$ tighten+combine

$\downarrow$

$$\langle 117, 73, 29, \ldots, 44 \rangle_x^{117} \leq 190x$$
$$190x \leq \langle 117, 73, 146 \ldots, 161 \rangle_x^{117}$$

$\Rightarrow$?

$\Rightarrow$ tighten+disjoin

$\downarrow$

$$1 \leq x \leq 0$$

$\Rightarrow$ combine

$\downarrow$

false

$\Rightarrow$ alternative: splintering by 117

$\longrightarrow$ $117 \leq 22230x' \leq 117$ / $\Rightarrow$ tighten+combine $\longrightarrow$ false

$\vdots$ (115 more)

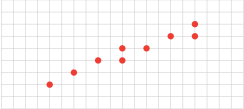
$\longrightarrow$ $-21996 \leq 22230x' \leq -21879$ / $\Rightarrow$ tighten+combine $\longrightarrow$ false

On $\mathbb{Q}$, Fourier-Motzkin Elimination can be used for projection
Omicron can as well, but produces a disjoint union.

$$x + 2 \leq 3y \leq x + 5$$
$$x - N + 1 \leq 2y \leq x + 1$$

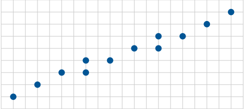
(x even)

$$4 \leq 2x' + 0 \leq 3N + \langle 4, 7 \rangle_N$$
$$1 \leq N$$



(x odd)

$$1 \leq 2x' + 1 \leq 3N + \langle 7, 4 \rangle_N$$
$$0 \leq N$$



A *decomposition* is a partition of a polyhedron such that,
in each part:

- each variable has a contiguous non-empty range
- with elementary bounds (no min / max)

To keep a collection of (disjoint) polyhedra: an AST

- **`if`** *condition* **`then`** *statements* [**`else`** *statements*]
  - arbitrary logical combinations
  - all inequalities properly tightened
  - no *mixed* inequalities (thanks to Disjoin)
- **`exec`** *label*
- **`for/when`** *PLE* **`<=`** *scale* × *counter* **`<=`** *PLE* *statements*
  - *scale* used *only* to keep bounds tight, e.g.,

```
for 2x + [x:6,4,8] <= 6y <= 3x + [x:0,3] ...
```

The AST keeps a layer for each variable. On LS(N), start with

```
when _ <= N <= _
    for _ <= x <= _
        for _ <= y <= _
            if 3y <= x+[x:3,5,4] and ... then
                exec S
```

Starting from an inequality and its innermost enclosing loop

for/when $L \leq sx_n \leq U$

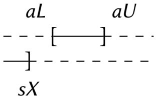
  … $ax_n \leq X$ …

Affine unswitching produces:

if $sX < aL$ then

  for $L \leq sx_n \leq U$ do

    … $ax_n \leq X$ … // = false



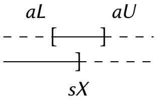
else if $sX < aU$ then

  for $aL \leq asx_n \leq sX$ do

    … $ax_n \leq X$ … // = true

  for $s(X + a) \leq asx_n \leq aU$ do

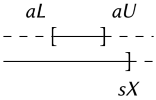
    … $ax_n \leq X$ … // = false



else // $sX \geq aU$

  for $L \leq sx_n \leq U$ do

    … $ax_n \leq X$ … // = true



Periodic inequalities need special treatment:

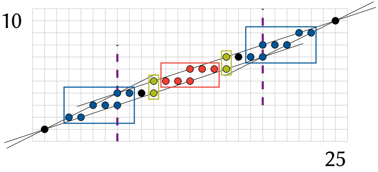
$$\langle X_0, X_1, \ldots \rangle_{x_n}^{\pi_n} \geq 0 \qquad \text{is viewed as} \qquad \langle X_0 \geq 0,\ X_1 \geq 0,\ \ldots \rangle_{x_n}^{\pi_n}$$

then individual inner inequalities are hoisted,
eventually leaving a *periodic boolean*:

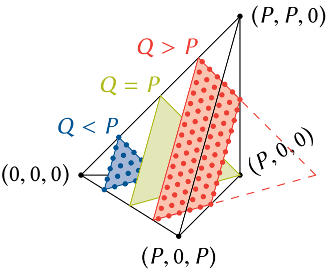
$$\langle b_0, b_1, \ldots \rangle_{x_n}^{\pi_n} \qquad \text{with } b_i \in \{\texttt{true}, \texttt{false}\}$$

At this point, the `for`-range on x_n is unrolled by a factor π_n

```
when N = 0
  exec S(1,1); exec S(3,2); exec S(5,3); exec S(7,4)
when N = 1 [...] when N = 2 [...] when N = 3 [...] when N = 4 [...]
when 5 <= N <= _
  exec S(1,1)
  for 3 <= x <= 8
      for 2x+[x:6,4,8] <= 6y <= 3x+[x:0,3]
          exec S(x,y)
  exec S(9,4)
  for 4 <= y <= 5
      exec S(10,y)
  for 11 <= x <= 3N-3
      for x+[x:3,2,4] <= 3y <= x+[x:3,5,4]
          exec S(x,y)
  for N <= y <= N+1
      exec S(3N-2,y)
  exec S(3N-1,N+1)
  for 3N <= x <= 3N+5
      for 3x-3N+[x:[N:6,3],[N:3,6]] <= 6y <= 2x+[x:6,10,8]
          exec S(x,y)
  exec S(3N+7,N+4)
```



$$\begin{cases} 0 \le i \le P \\ 0 \le j \le i \\ 0 \le k \le i - j \\ Q = i + j + k \end{cases}$$




```
when P = 0
  when Q = 0
      exec S(0,0,0)
when 1 <= P <= _
  when 0 <= Q <= P
      for Q+[Q:0,1] <= 2i <= 2Q
          for 0 <= j <= -i+Q
              exec S(i,j,-j-i+Q)
  when P+1 <= Q <= 2P
      for Q+[Q:0,1] <= 2i <= 2P
          for 0 <= j <= -i+Q
              exec S(i,j,-j-i+Q)
```

Most polyhedral operations can be implemented by hoisting:

- image (and pre-image): e.g., skewing a rectangle

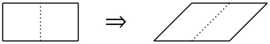
```
for _ <= x <= _
    for _ <= y <= _
        if 0 <= x <= 19 and 0 <= y <= 9 then

            exec S(x,y)
```

Most polyhedral operations can be implemented by hoisting:

- image (and pre-image): e.g., skewing a rectangle

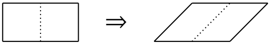
```
for _ <= x' <= _
  for _ <= y' <= _
    for _ <= x <= _
      for _ <= y <= _
        if 0 <= x <= 19 and 0 <= y <= 9 then
          if x' = x+y and y' = y then
            exec S(x',y',x,y)
```



Most polyhedral operations can be implemented by hoisting:

- image (and pre-image): e.g., skewing a rectangle

```
for _ <= x' <= _
  for _ <= y' <= _
    for _ <= x <= _
      for _ <= y <= _
        if 0 <= x <= 19 and 0 <= y <= 9 then
          if x' = x+y and y' = y then
            exec S(x',y',x,y)
```

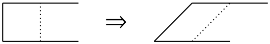


After unswitching:

```
for 0 <= x' <= 9
    for 0 <= y' <= x'
        exec S(x',y',-y'+x',y')
for 10 <= x' <= 19
    for 0 <= y' <= 9
        exec S(x',y',-y'+x',y')
for 20 <= x' <= 28
    for x' - 19 <= y' <= 9
        exec S(x',y',-y'+x',y')
```

Most polyhedral operations can be implemented by hoisting:

- image (and pre-image): e.g., skewing a rectangle



```
for _ <= x' <= _
  for _ <= y' <= _
    for _ <= x <= _
      for _ <= y <= _
        if 0 <= x <= 19 and 0 <= y <= 9 then
          if x' = x+y and y' = y then
            exec S(x',y',x,y)
```

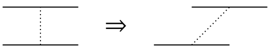
After unswitching:

```
for 0 <= x' <= 9
    for 0 <= y' <= x'
        exec S(x',y',-y'+x',y')
for 10 <= x' <= 19
    for 0 <= y' <= 9
        exec S(x',y',-y'+x',y')
for 20 <= x' <= 28
    for x' - 19 <= y' <= 9
        exec S(x',y',-y'+x',y')
```

Most polyhedral operations can be implemented by hoisting:

- image (and pre-image): e.g., skewing a rectangle

```
for _ <= x' <= _
  for _ <= y' <= _
    for _ <= x <= _
      for _ <= y <= _
        if 0 <= x <= 19 and 0 <= y <= 9 then
          if x' = x+y and y' = y then
            exec S(x',y',x,y)
```



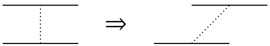
After unswitching:

```
for 0 <= x' <= 9
    for 0 <= y' <= x'
        exec S(x',y',-y'+x',y')
for 10 <= x' <= 19
    for 0 <= y' <= 9
        exec S(x',y',-y'+x',y')
for 20 <= x' <= 28
    for x' - 19 <= y' <= 9
        exec S(x',y',-y'+x',y')
```

Most polyhedral operations can be implemented by hoisting:

- image (and pre-image): e.g., skewing a rectangle

```
for _ <= x' <= _
  for _ <= y' <= _
    for _ <= x <= _
      for _ <= y <= _
        if 0 <= x <= 19 and 0 <= y <= 9 then
          if x' = x+y and y' = y then
            exec S(x',y',x,y)
```



After unswitching:

```
for 10 <= x' <= 19
    for 0 <= y' <= 9
        for x'-y' <= x <= x'-y'
            for y' <= y <= y'
                exec S(x',y',-y'+x',y')
```

After repeated hoisting/unswitching:

- no `if-then-else` conditional parts
- no empty range

→ lexicographic extrema are readily available

```
for 0 <= x' <= 9
    for 0 <= y' <= x'
        exec S(x',y',-y'+x',y')
for 10 <= x' <= 18
    for 0 <= y' <= 9
        exec S(x',y',-y'+x',y')
for 19 <= x' <= 28
    for x' - 19 <= y' <= 9
        exec S(x',y',-y'+x',y')
```

Linear optimization: min/maximize $\boxed{15z + 2 = x}$ over $\mathrm{LS}(N)$

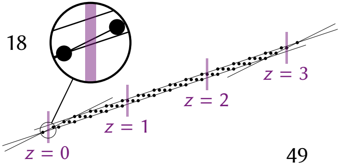
```
when _ <= N <= _

    for _ <= x <= _
        for _ <= y <= _
            if ... then

                exec S
```

Linear optimization: min/maximize $\boxed{15z + 2 = x}$ over LS(N)

```
when _ <= N <= _
  for _ <= z <= _
    for _ <= x <= _
      for _ <= y <= _
        if ... then
          if 15z+2 = x then
            exec S
```



Linear optimization: min/maximize $\boxed{15z + 2 = x}$ over $\mathrm{LS}(N)$

```
when _ <= N <= _
  for _ <= z <= _
    for _ <= x <= _
      for _ <= y <= _
        if ... then
          if 15z+2 = x then
            exec S
```

produces

```
when N = 4
    exec S(1,17,7)
when 5 <= N <= _
    for 5 <= 5z <= N+[N:0,-1,-2,-3,1]
        exec S(z,15z+2,5z+2)
```

i.e., $z_{\mathrm{min}} = 1$ (at $x = 17$) and $z_{\mathrm{max}} = \frac{N - \langle 0,1,2,3,-1\rangle_N}{5} = \lceil \frac{N-3}{5} \rceil = \lfloor \frac{N+1}{5} \rfloor$

Decomposition / Finite State Machines

States: one per (static) exec statement

Transitions: given by function NEXT (→), defined with FIRST (→):

```
[...]
for 3 <= x <= 10 do
    for 2x+[x:6,4,8] <= 6y' <= 3x+[x:0,3] do
        exec S2(x,y')
    done
done

for 11 <= x <= 3N-3 do
    for x+[x:3,2,4] <= 3y' <= x+[x:3,5,4] do
        exec S3(x,y')
[...]
```

Note: the result of NEXT tests each variable exactly once ("at" done)

Summary

- A new representation for inequalities
- Tightening
- Precise combination/comparison
- A correct and complete decision procedure
- Polyhedron decomposition into simple ranges
- Essential polyhedral operations reformulated

More work needed on:

- reducing size/complexity of representations and algorithms
  - delay normalization of “deeper levels”
  - leverage more arithmetic properties
- strategies & heuristics for “simplest” decomposition
  - very frequent excessive fragmentation
  - avoidance or correction?

The modulo of a PLE is a PLE

$$\begin{aligned} & \left(a_n x_n + \langle X_0, \ldots \rangle_{x_n}^{\pi_n}\right) \bmod \beta \\ = \;& \langle 0, 1, \ldots \rangle^{\beta}_{a_n x_n + \langle X_0, \ldots \rangle_{x_n}^{\pi_n}} \\ = \;& {}^{k}\Big\langle \ldots, \quad \left(a_n k + X_{(k \bmod \beta)}\right) \bmod \beta \quad \ldots \Big\rangle_{x_n}^{\pi'_n} \\ & \text{where } \pi'_n = \mathrm{lcm}(\pi_n, \beta / \gcd(a_n, \beta)) \end{aligned}$$

The overall size of a corrective term for

$$a_n x_n \leq \langle X_0, \ldots \rangle_{x_n}^{\pi_n}$$

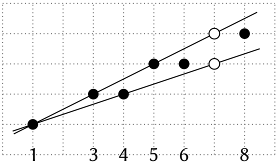
is

$$\prod_{i=1}^{n} \mathrm{lcm}(\pi_n, \frac{\beta}{\gcd(a_n, \beta)})$$

Mixed tight bounds are “fuzzy”

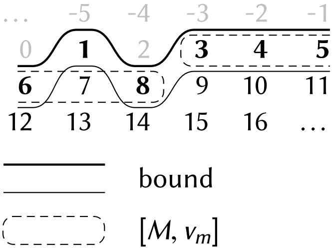
$$x + \langle 3, 2, 4\rangle_x \leq 3y \quad 2y \leq x + \langle 0, 1\rangle_x$$

$$\Rightarrow \quad \langle 6, 1, 8, 3, 4, 5\rangle_x \leq x$$



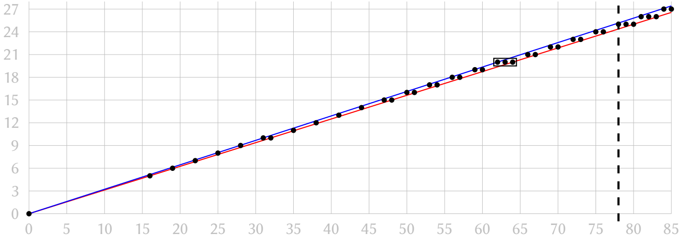
These can be turned into disjunctions of *linear* bounds

DISJOIN_1($\langle v_0, \ldots\rangle_x^\pi \leq ax$)
  let $v_m = \max\{v_i\}$
  let $M = v_m - a(\pi - 1)$
  let $\mathcal{O} = \{v_i \mid v_i < M\}$
  return $(M \leq ax) \vee$
    $(\bigvee_{d\in\mathcal{O}}(x = d))$



Provides a *major* bound (M) plus *outliers* ($\mathcal{O}$)
$\Rightarrow (x = 1) \vee (3 \leq x)$

$$31y \le 10x \le 32y$$
$$\Rightarrow \quad 10x \le 32y \;\wedge\; 31y \le 10x$$
$$\Rightarrow \quad \langle 0, 497, 498, \ldots, 493, 494, 495 \rangle_x^{496} \le x$$



$$x = 0 \vee x = 16 \vee x = 19 \vee x = 22 \vee x = 25 \vee x = 28$$
$$\vee \; 31 \le x \le 32 \;\vee\; x = 35 \vee x = 38 \vee x = 41 \vee x = 44$$
$$\vee \; 47 \le x \le 48 \;\vee\; 50 \le x \le 51 \;\vee\; 53 \le x \le 54$$
$$\vee \; 56 \le x \le 57 \;\vee\; 59 \le x \le 60 \;\vee\; \boxed{62 \le x \le 64}$$
$$\vee \; 66 \le x \le 67 \;\vee\; 69 \le x \le 70 \;\vee\; 72 \le x \le 73$$
$$\vee \; 75 \le x \le 76 \;\vee\; \mathbf{78 \le x}$$

Multidimensional mixed bounds rely on transposition, e.g.:

$$x \leq 3N + \begin{pmatrix} \langle 0,3\rangle_N, \langle 7,4\rangle_N, \langle 2,5\rangle_N, \\ \langle 3,0\rangle_N, \langle 4,7\rangle_N, \langle 5,2\rangle_N \end{pmatrix}^6_x$$

1. Build the uni-dimensional bound
   for all phases of all other variables

$$x \leq 3N + \langle \quad \langle 0,7,2,3,4,5\rangle_x, \quad \langle 3,4,5,0,7,2\rangle_x \quad \rangle_N$$

2. Apply Disjoin_1 on each "sub-bound"
   to obtain the major bound and outliers

$$x \leq 3N + \langle \quad [\![M_0 = 5, \mathcal{O}_0 = \{7\}]\!], \quad [\![M_1 = 5, \mathcal{O}_1 = \{7\}]\!] \quad \rangle_N$$

3. Collect phase-specific major bounds and outliers
   into periodic numbers

$$(x \leq 3N + \langle 5,5\rangle_N) \vee (x = 3N + \langle 7,7\rangle_N)$$

(+ simplify, + other details)

With multidimensional bounds $\langle X_0, \ldots \rangle_{x_n}^{\pi_n} \leq a_n x_n$

- transpose to "sink" x_n at the lowest level
- apply Disjoin_1 (for each phase of each other variable)
- transpose "back" the results

$$\begin{bmatrix}\begin{bmatrix}X_{00}\\ X_{01}\\ \vdots\end{bmatrix}_{x_{n-1}}^{\pi_{n-1}}, \begin{bmatrix}X_{10}\\ X_{11}\\ \vdots\end{bmatrix}_{x_{n-1}}^{\pi_{n-1}}, \ldots\end{bmatrix}_{x_n}^{\pi_n} \xrightarrow{\text{(a)}} \begin{bmatrix}[X_{00}, X_{10}, \ldots]_{x_n}^{\pi_n}\\ [X_{01}, X_{11}, \ldots]_{x_n}^{\pi_n}\\ \vdots\end{bmatrix}_{x_{n-1}}^{\pi_{n-1}}$$

$$\text{(b)}$$

$$\left\{\begin{bmatrix}M_0\\ M_1\\ \vdots\end{bmatrix}_{x_{n-1}}^{\pi_{n-1}}, \begin{bmatrix}O_{01}\\ O_{11}\\ \vdots\end{bmatrix}_{x_{n-1}}^{\pi_{n-1}}, \ldots\right\} \xleftarrow[\text{(c)}]{} \begin{bmatrix}\left\{M_0, O_{01}, \ldots\right\}\\ \left\{M_1, O_{11}, \ldots\right\}\\ \vdots\end{bmatrix}_{x_{n-1}}^{\pi_{n-1}}$$